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A Stochastic Theory of the Firm

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Abstract

We present a stochastic model of make-to-stock firms based on a buffer flow system with jumps. The cumulative production and the cumulative demand are governed by two Poisson counting processes with random intensities parameterized by production capacity and price respectively. Optimal operating and pricing policies (short-run decisions) and optimal capacity (long-run) decisions are explored by application of a two-stage optimization device. Detailed computations regarding the Poisson buffer flow system and a variation on the basic model with learning effects are also presented.

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1. Introduction

This article develops a stochastic model of a make-to-stock firm in a continuous time framework. The model, on the one hand, generalizes the classical economic theory of monopoly to include the dynamic aspects in the presence of both demand and production uncertainties. See [2], [3], [8], [15], [19], [24] and [25] for this line of literature. All the models studied in the above articles have a single-period or a multiperiod setting with demand uncertainty only. De Vany introduces a queueing model of make-to-order firms in [6] and uses long-run average profit criterion as the objective. On the other hand, our basic model also parallels the type of models in inventory theory, but expands the framework to explicitly consider capacity and price as decision variables. In particular, the buffer system of continuous flows discussed in Harrison [11] is extended to the flow systems with jumps in which the explicit consideration of capacity and price decisions is possible.

The basic assumption of our model is that the cumulative production and cumulative demand are two (Poisson) counting processes parameterized by production capacities and prices respectively. Mimicking the real life operation, the firm's decision is two fold: it makes a static design decision (capacity decision) at time zero, and exercises the dynamic control of production rate and price over time. A total discounted profit criterion is used. The optimal capacity decision and the optimal operating and pricing policies are explicitly characterized via a two-stage optimization procedure. That is, we first find an optimal policy for each given capacity level, and then select a capacity level to maximize the profit function obtained in the first step assuming that whatever the capacity level is selected, the firm operates optimally thereafter.

The article is arranged as follows. Section 2 is devoted to the formulation of the basic model. Though we make a quite restrictive Poisson assumption on the production and demand processes, the formulation readily extends to more general additive processes. In Section 3, we consider a special case in which the firm sets the price at the beginning and keeps it unchanged over time. The condition for an optimal barrier policy and the value function under a barrier policy are explicitly computed by the strong Markov and renewal properties of the inventory process and an argument justifying the optimality of the barrier policy is provided. Section 4 discusses the basic model with dynamic pricing and its economic implications. In Section 5, we study an interesting variation on the basic model where achievable capacity expands with cumulative production due to learning. Concluding remarks follow in Section 6.

2. Formulation

We shall consider a firm that continually produces and sells a single commodity. The product is placed in an inventory buffer where it is taken out to satisfy demand. Demands that occur while the inventory level is zero are not filled. The cumulative input (production) and output (fulfilled demand) are represented by the increasing non-negative integer-valued stochastic processes

$A = \{A(t), t \geq 0\}$ and $B = \{B(t), t \geq 0\}$, where $A(t)$ and $B(t)$ denote the amount of production and the amount of demand fulfilled in the time interval $(0, t]$. Then the inventory level at time t is

$$Z(t) = x + A(t) - B(t), \tag{2.1}$$

where x is the amount of inventory at time zero. We assume that

A, B are independent Poisson processes with random intensities $\{\alpha_t, t \geq 0\}$ and $\{\beta_t, t \geq 0\}$. (2.2)

That is, $\{A(t) - \int_0^t \alpha_s ds, t \geq 0\}$ and $\{B(t) - \int_0^t \beta_s ds, t \geq 0\}$ are martingales. In other words, $\{\int_0^t \alpha_s ds, t \geq 0\}$ and $\{\int_0^t \beta_s ds, t \geq 0\}$ are compensators of the counting processes A and B respectively. The only real requirement here is that processes α and β be integrable and be adapted (in the sense to be explained shortly).

To have a model that embraces pricing and facility design decisions, one must have stochastic analogs for the firm's demand and cost functions. That is, we need a family of demand processes B parameterized by prices, and a similarly parameterized family of production processes A. For the production process, let $\alpha(> 0)$ be the average output rate when the firm working at its full capacity. The value α is a function of designed capacity factors such as capital and labor invested. We use α as a primitive parameter which measures the capacity of the firm which is determined at the beginning. Viewing α_t as the actual production rate the firm employs at time t, we require that $0 \leq \alpha_t \leq \alpha$ for all $t \geq 0$. Note that if $\alpha_t = \alpha$ for all $t \geq 0$, then A becomes the potential input process which is Poisson with rate α. Also, we assume the firm can control the average demand rate β through pricing and the inverse demand function $p(\beta)$ is a deterministic decreasing function. The potential revenue rate $p(\beta)\beta$ is assumed to be concave as usual. Management is also assumed to be free to reject potential sales. That means the firm may set the demand rate to be zero whenever it is willing or forced to do so.

Given the inverse demand function $p(\cdot)$ and the capacity α (selected at time zero), a *feasible operating policy* is defined as a pair of stochastic processes (α_t, β_t) that jointly satisfy the following:

(2.3) (α_t) and (β_t) are left continuous and have right-hand limits.

(2.4) (α_t) and (β_t) are adapted with respect to Z,

(2.5) $0 \leq \alpha_t \leq \alpha, \beta_t \geq 0$ and β_t is bounded, for all $t \geq 0$,

(2.6) $Z(t)$ is non-negative for all t.

Condition (2.4) implies that α_t and β_t are functions of $(Z(s), s \leq t)$. This says that the control that the firm exercises at time t is based on only the historical information before time t. Conditions (2.3) and (2.5) ensure that (α_t) and (β_t) are integrable and predictable with respect to Z. The restriction (2.6) implies that backlogging is not allowed: sales which cannot met from stock on hand are simply lost, and has no effect on future demand.

For a fixed capacity and a fixed price, our model parallels the buffered flow system described in [11], but it allows the system flow be represented by jump processes. One way to justify that there exist processes Z, A and B is to generalize Harrison's proofs in Chapter 2 of [11] by assuming the primitive netput process X is in $D[0,\infty)$, the space of real-valued functions on $\Re_+$ that are right continuous with left limits, instead of $C[0,\infty)$, the space of real-valued continuous functions on $\Re_+$. We shall show a sample path construction of processes Z, A and B for a simple flow system to illustrate that there is no difficulty to make such generalization.

Consider a flow system with infinite buffer capacity in which $\alpha_t = \alpha > 0, \beta_t = \beta > 0$ if $Z(t) > 0$ and $\beta_t = 0$ if $Z(t) = 0$. We take $X(0)$ as the initial inventory level, $\hat{A}(t)$ as the cumulative input up to time t, and $\hat{B}(t)$ as the cumulative *potential* output up to time t. Denote by $L(t)$ the amount of potential output lost up to time t because of the buffer emptiness. So the *actual* input $A(t)$ is $\hat{A}(t)$ and the *actual* output $B(t)$ is $\hat{B}(t) - L(t)$ over $[0,t]$. Setting

$$X(t) \equiv X(0) + \hat{A}(t) - \hat{B}(t). \tag{2.7}$$

the inventory at time t is then given by

$$Z(t) \equiv X(0) + A(t) + B(t) = X(t) + L(t). \tag{2.8}$$

We require the lost potential output process L satisfy

L is increasing and $RCLL$ with $L(0) = 0$, and

L increases only when $Z = 0$, to be consistent with the physical restriction $Z(t) \geq 0$ for all $t \geq 0$.

We will show that the above conditions uniquely determine L and further imply

$$L(t) = \sup_{0 \leq s \leq t} X^-(s).$$

That is, L can be concisely represented in terms of the primitive process X. The proof involves construction of sample paths only.

Denote by $x = (x_t, t \geq 0)$, the generic element of D. Define mappings $\psi, \phi : D \to D$ by

$$\psi_t(x) \equiv \sup_{0 \leq s \leq t} x_s^- \text{ and}$$

$$\phi_t(x) \equiv x_t + \psi_t(x)$$

for $t \geq 0$. Fix $x \in D$ and let $l \equiv \psi(x)$ and $z \equiv \phi(x) = x + l$. Then z is obtained from x by imposition of a lower control barrier at zero.

Proposition 2.1. *Suppose $x \in D$ and $x_0 \geq 0$. Then $\psi(x)$ is the unique function l such that*

(2.9) l *is right continuous with left hand limits and increasing with* $l_0 = 0$.

(2.10) $z_t = x_t + l_t \geq 0$ *for all* $t \geq 0$ *and*

(2.11) l *increases only when* $z = 0$.

Proof. Fix $x \in D$ and $l \equiv \psi(x)$ and $z \equiv x + l$. It can be verified that this l does satisfy (2.9)-(2.11). To prove uniqueness, let l^* be any other solution of (2.9)-(2.11) and set $z^* \equiv x + l^*$. Setting $y \equiv z^* - z = l^* - l$, we note that y is a $RCLL$ and VF (variation finite) function with $y_0 = 0$. Since y has at most countable many discontinuity points of first type, we enumerate them as $t_1, t_2, \ldots$. Suppose $y_{t_{i-1}} = 0$. Note that y is continuous on the interval $[t_{i-1}, t_i)$ and hence $y_t = 0$ for $t \in [t_{i-1}, t_i)$ by proposition 2.3 in [11]. Then

$$\begin{aligned}(z^*_{t_i} - z_{t_i})^2 &= (z^*_{t_i} - z_{t_i})^2 - (z^*_{t_i-} - z_{t_i-})^2 \\ &= [(l^*_{t_i} - l^*_{t_i-}) - (l_{t_i} - l_{t_i-})](z^*_{t_i} - z_{t_i}) \\ &= (z^*_{t_i} - z_{t_i})(l^*_{t_i} - l^*_{t_i-}) + (z_{t_i} - z^*_{t_i})(l_{t_i} - l_{t_i-})\end{aligned} \tag{2.12}$$

since $y_{t_i-} = z^*_{t_i-} - z_{t_i-} = l^*_{t_i-} - l_{t_i-} = 0$. We know that l^* increases only when $z^* = 0$, and $z \geq 0$, so the first term on the right side of the last equation is ≤ 0, and identical reasoning shows that the second term is ≤ 0 as well. But the left side of (2.12) is ≥ 0, so $y_{t_i} = z^*_{t_i} - z_{t_i} = 0$. So far, we have proved that if $y_{t_i} = 0$, we can conclude $y_t = 0$ for $t \in [t_i, t_{i+1}]$.

The only thing left to show is that $y_{t_i} = 0$ for all $i \in N$ where $N \equiv \{1, 2, \ldots\}$. Let $I \equiv \{i : y_{t_i} \neq 0\}$ and $t^* \equiv \inf\{t_i : i \in I\}$. We want to show that $I = \emptyset$ or, equivalently, $t^* = \infty$. First, note that $N \setminus I \neq \emptyset$ since $y_0 = 0$ implies $y_{t_1} = 0$. Suppose $t^* < \infty$. If $t^* = t_{i^*}$ for some $i^* \in I$, then $y_{t_{i^*-1}} = 0$ implies $y_{t_{i^*}} = 0$, a contradiction. Thus $t^* < t_i$ for all $i \in I$. Now let $t_{j^*} \equiv \sup\{t_j : t_j \leq t^*\}$. Certainly $t_{j^*+1} > t^*$, and $j^* + 1 \notin I$ since $y_{t_{j^*}} = 0$ implies $y_{t_{j^*+1}} = 0$. But $t_{j^*+1} < t_i$, for all $i \in I$. This contradicts the fact that t^* is the infimum. Thus, $t^* = \infty$ or I is empty.

Therefore $y = 0$, hence $l^* = l$, and the proof is complete

∎

The other results in Chapter 2 of [11] can be similarly generalized to the case that $x \in D$. It is noteworthy that the proof of the above proposition does not use the assumption that x is the trajectory of the difference of two Poisson processes, and hence the result remain valid as long as Ω is space D. Though we consider a special case of Poisson, the results in the paper are not hard to extend to the case of compound Poisson processes, or more general buffer system of additive processes.

To complete our formulation, we specify the cost structure as follows. The firm incurs a fixed cost $C(\alpha)$ at time zero, which is used to build capacity α, and $C(\alpha)$ is increasing in α. The firm also incurs a linear variable cost, say c dollars per unit of actual production. A cost of h dollars

per unit time is incurred for each unit of production held in inventory. Therefore the expected discounted revenue over the infinite horizon is

$$TR(x) \equiv E_x \left\{ \int_0^\infty e^{-rt} p(\beta_t) dB(t) \right\}.$$

where $r > 0$ is a discount factor. The expected cost is

$$TC(x) \equiv E_x \left\{ \int_0^\infty e^{-rt} [c dA(t) + hZ(t)dt] \right\} + C(\alpha).$$

The objective of the firm is to choose a production capacity α, and a pair of control processes (α_t, β_t) to maximize the expected discounted profit

$$\Pi(x) = TR(x) - TC(x) \tag{2.13}$$

such that assumptions (2.1)-(2.2), and feasibility constraints (2.3)-(2.6) are satisfied. The following proposition transforms objective function (2.13) to an equivalent form which is easy to deal with in finding the optimal operating policies.

Proposition 2.2. *For any given policy* (α_t, β_t),

$$\Pi(x) = V(x) - \frac{h}{r} \cdot x - C(\alpha), \tag{2.14}$$

where

$$V(x) \equiv E_x \left\{ \int_0^\infty e^{-rt} [(p(\beta_t) + \frac{h}{r}) dB(t) - (c + \frac{h}{r}) dA(t)] \right\}. \tag{2.15}$$

Furthermore,

$$V(x) = E_x \left\{ \int_0^\infty e^{-rt} [(p(\beta_t) + \frac{h}{r})\beta_t + (c + \frac{h}{r})\alpha_t] dt \right\}. \tag{2.16}$$

Proof. Equation (2.14) can be obtained by substituting the inventory equation (2.1) into (2.13) and applying the integration by parts formula for the Riemann-Stieljes integrals. Equation (2.16) follows from the assumption that A and B are Poisson processes with random intensity $\{\alpha_t\}$ and $\{\beta_t\}$ and the fact that the integrands in (2.15), $e^{-rt}(p(\beta_t) + \frac{h}{r})$ and $e^{-rt}(c + \frac{h}{r})$, are left-continuous and right-limited processes adapted to Z, hence predictable.

∎

The proposition says that with each unit produced to stock, the firm actually incurs a cost c and an opportunity loss $\frac{h}{r}$ if this unit would be stocked forever; with each unit sold from stock, the firm's actual gain would be selling price p plus an opportunity gain $\frac{h}{r}$ that is equal to the

opportunity loss if this unit would otherwise be stocked forever. The value $V(x)$ can be considered as the gross profit incurred by the operations after time zero.

The firm's problem involves a two-stage optimization. The first stage involves capacity selection for the plant. The second stage is to find the optimal operating policy, the production decisions as well as pricing decisions. To solve the problem, we proceed reversely, i.e., first find a unique optimal operating policy for each given capacity level α, and then select a capacity α to maximize the profit value functions obtained in the first step assuming the firm operates optimally whatever the capacity level is set. This two-stage optimization approach is the procedure that we shall follow throughout this article.

3. Pricing As A Design Decision

First, let us assume that the firm resolves its price decision at time zero and the price remain unchanged over time. Then condition (2.4) reduces to

$$0 \leq \alpha_t \leq \alpha, \beta_t \in \{0, \beta\}, \text{ for all } t \geq 0. \tag{3.1}$$

where β is a design variable. The firm's problem is to select a selling price p (equivalently a potential demand rate β), a production capacity α at the beginning, and a pair of control processes (α_t, β_t) to maximize profit Π. The value function defined in (2.15) is of the form

$$V(x) = E_x \left\{ \int_0^\infty e^{-rt}(q\beta_t - w\alpha_t)dt \right\}. \tag{3.2}$$

where $q \equiv p + \frac{h}{r}$, and $w \equiv c + \frac{h}{r}$. Assume $p > c$, and then $q > w$. Let's first consider a class of feasible policies, namely, *barrier policies*. A barrier policy is, for some $b > 0$, $\alpha_t = \alpha 1_{[0,b)}(Z(t-))$ and $\beta_t = \beta 1_{(0,b]}(Z(t-))$, for $t \geq 0$. This means that production is always at full capacity except that it ceases if the inventory reaches level b and resumes when the inventory is depleted by one unit. Demands are rejected only when products are unavailable in stock. This is really a simple policy since it depends on only one number b if all other parameters are fixed. In fact, under a barrier policy with parameter $b > 0$, the inventory constant process Z is a Markov process with state space $E = \{0, 1, \ldots, b\}$. Specifically, it is a birth and death process with finite state space or a $M/M/1/b$ queue. The following computations have been derived in Li [16] and we simply list them without proofs.

Denote the first time at which state y is reached and its Laplace transform by

$$T(y) \equiv \inf\{t \geq 0, Z(t) = y\}, \text{ and}$$

$$\theta(x, y) \equiv E_x\left[e^{-rT(y)}\right]$$

. Then

Lemma 3.1. *For $0 \le x \le b$,*

$$\theta(x,0) = \frac{g(b-x)}{g(b)}, \ \text{and} \ \theta(x,b) = \frac{e(x)}{e(b)}. \tag{3.3}$$

where

$$g(x) \equiv (\rho_2 - 1)\rho_1^{-x} - (\rho_1 - 1)\rho_2^{-x},$$
$$e(x) \equiv (\rho_2^{-1} - 1)\rho_1^{x} - (\rho_1^{-1} - 1)\rho_2^{x},$$

and ρ_1 and ρ_2 are the two roots of the quadratic equation

$$\rho = \frac{\alpha}{\alpha+\beta+r}\cdot\rho^2 + \frac{\beta}{\alpha+\beta+r} \tag{3.4}$$

with $0 < \rho_1 < 1, \rho_2 > 1$, for $r > 0$. Furthermore, $g(\cdot)$ is strictly increasing, $e(\cdot)$ is strictly decreasing, hence, $\theta(b,0)$ is strictly decreasing in b and $\theta(\infty,0) = 0$.

Following (3.2), the value function under a barrier policy with parameter b can be written as

$$\begin{aligned} V(x) &= E_x\left\{\int_0^\infty e^{-rt}[q\beta 1_{(0,b]}(Z(t)) - w\alpha 1_{[0,b)}(Z(t))]dt\right\} \\ &= q\cdot\frac{\beta}{r} - w\cdot\frac{\alpha}{r} + \hat{V}(x), \end{aligned} \tag{3.5}$$

where

$$\hat{V}(x) = w\cdot E_x\left\{\int_0^\infty e^{-rt}\alpha 1_{\{b\}}(Z(t))dt\right\} - q\cdot E_x\left\{\int_0^\infty e^{-rt}\beta 1_{\{0\}}(Z(t))dt\right\}. \tag{3.6}$$

The term $E_x\left\{\int_0^\infty e^{-rt}\alpha 1_{\{b\}}(Z(t))dt\right\}$ can be thought as the expected total discounted potential output lost due to ceasing production when the inventory level reaches the limit b, while the term $E_x\left\{\int_0^\infty e^{-rt}\beta 1_{\{0\}}(Z(t))dt\right\}$ is the expected total discounted potential sales lost due to stockout. And $\hat{V}$ is the profit gain, in expected present value terms, under the barrier policy over that of the potential production and demand. By the strong Markov and the renewal properties of process Z, we have

Lemma 3.2.

$$E_x\left\{\int_0^\infty e^{-rt}\beta\cdot 1_{\{0\}}(Z(t))dt\right\} = \frac{\beta}{\alpha+r}\cdot\frac{\theta(x,0)}{1-\frac{\alpha}{\alpha+r}\theta(1,0)}, \ \text{and} \tag{3.7}$$

$$E_x\left\{\int_0^\infty e^{-rt}\alpha\cdot 1_{\{b\}}(Z(t))dt\right\} = \frac{\alpha}{\beta+r}\cdot\frac{\theta(x,b)}{1-\frac{\beta}{\beta+r}\theta(b-1,b)}. \tag{3.8}$$

Thus the explicit form of $\hat{V}$, hence V, can be obtained. We shall denote by $\hat{V}^b$ (V^b) the value function under a barrier policy with a upper barrier b if it is necessary to do so.

Proposition 3.1. *There exist an optimal barrier policy with one critical inventory limit b^* that is uniquely determined by the condition*

$$\theta(b^* + 1, 0) \leq \frac{w}{q}, \text{ and } \theta(b^*, 0) > \frac{w}{q}. \tag{3.9}$$

Proof. For a fixed x,

$$V^b(x) - V^{b-1}(x) = \hat{V}^b(x) - \hat{V}^{b-1}(x) = G \cdot \left(\theta(b, 0) - \frac{w}{q}\right), \tag{3.10}$$

where

$$G \equiv \frac{q\beta\rho_1^b\rho_2^b(-e(x))g(b)}{r(\rho_2^{b+1} - \rho_1^{b+1})(\rho_2^b - \rho_1^b)} > 0, \text{ for } b \geq 1$$

since all the terms in the numerator and the denominator are positive. So the sign of $V^b(x) - V^{b-1}(x)$ is exactly the same as that of $\theta(b, 0) - \frac{w}{q}$. Since $\theta(\cdot, 0)$ is strictly decreasing, the optimal b^* should be the one such that $V^{b^*+1}(x) - V^{b^*}(x) \leq 0$ and $V^b(x) - V^{b-1}(x) \geq 0$ which is equivalent to condition (3.9)

∎

We now want to show that this optimal barrier policy is indeed optimal among all feasible policies. In other words, the class of Markovian controls in our setting is complete.

Under a barrier policy, if the inventory level starts from the set of states $E = \{0, 1, \ldots, b\}$, it will never get out again. We extend the value function with initial state $x(> b)$ by

$$V(b + k) = V(b) + kw, \text{ for } k > 0,$$

meaning that under a barrier policy, the extra k units would never have produced and the opportunity loss to have them produced is $w \equiv c + \frac{h}{r}$. Similarly define

$$V(-k) = V(0) - kq, \text{ for } k > 0.$$

Denote by V^* the value function under the optimal barrier policy, and $\Delta V(x)$ the difference $V(x) - V(x-1)$. We record the following two lemmas which can be proved by direct verification.

Lemma 3.3. *The V^* is strictly increasing and concave with $\Delta V^*(b+1) = w$ and $\Delta V^*(0) = q$.*

Define the operator Γ by

$$\Gamma f(x) \equiv \alpha f(x+1) + \beta f(x-1) - (\alpha + \beta) f(x),$$

where $f(\cdot)$ is an function on integer values. Then

Lemma 3.4. *The function $\hat{V}$ satisfies the difference equations*

$$\Gamma\hat{V}(x) - r\hat{V}(x) = 0, \text{ for } 0 \leq x \leq b-1,$$

and

$$\Gamma\hat{V}(x) - r\hat{V}(x) \leq 0, \text{ for } x \geq b.$$

One more technical lemma is needed in the verification of optimality.

Lemma 3.5. *Suppose $f(\cdot)$ is a function on integers. Then*

$$E_x[e^{-rt}f(Z(t))] = f(x)+E_x\left\{\int_0^t e^{-rs}[\alpha_s f(Z(s)+1) + \beta_s f(Z(s)-1) - (\alpha_s + \beta s + r)f(Z(s))]ds\right\}. \tag{3.11}$$

Proof. By integration by parts theorem, we have

$$\int_0^t e^{-rs}df(Z(s)) = e^{-rt}f(Z(t)) - f(Z(0)) + r\int_0^t e^{-rs}f(Z(s))ds. \tag{3.12}$$

Note that

$$\begin{aligned} E_x[\int_0^t e^{-rs}df(Z(s))] &= E_x\{\int_0^t e^{-rs}[(f(Z(s-)+1) - f(Z(s-)))dA(s) \\ &\quad + (f(Z(s-)-1) - f(Z(s-)))dB(s)]\} \\ &= E_x\{\int_0^t e^{-rs}[\alpha_s f(Z(s)+1) + \beta_s f(Z(s)-1) \\ &\quad - (\alpha_s + \beta_s)f(Z(s))]ds\}. \end{aligned} \tag{3.13}$$

The second equation follows from assumption (2.2) and that $e^{-rs}[f(Z(s-)+1) - f(Z(s-))]$ and $e^{-rs}[f(Z(s-)-1) - f(Z(s-))]$ are both left continuous, right-limited and adapted with respect to Z. Taking E_x of the both sides of (3.12) noticing $Z(0) = x$, substituting in the expression obtained in (3.13) and collecting the terms, we conclude the proof

∎

Note that as a special case of a barrier policy, Γ is the generator of Z and Lemma 3.5 implies that

$$e^{-rt}f(Z(t)) - \int_0^\infty e^{-rs}(\Gamma f - rf)(Z(s))ds \text{ is a martingale.}$$

Here is the main result of the section.

Proposition 3.2. *The barrier policy with inventory limit uniquely determined by condition (3.9) is optimal among all the feasible policies.*

Proof. Denote by F the value function with arbitrary feasible policy (α_t, β_t). Define

$$V_t(x) \equiv E_x[\int_0^t e^{-rs}(q\beta_s - w\alpha_s)ds + e^{-rt}V^*(Z(t))], \tag{3.14}$$

for $t > 0$ and $x \geq 0$. This is the value function for a hybrid policy that follows (α_s, β_s) up to time t, yielding a inventory content of $Z(t)$, and then enforces the optimal barrier policy with value function V^* thereafter. By (3.11),

$$\begin{aligned} E_x[e^{-rt}V^*(Z(t))] &= V^*(x) + E_x\{\int_0^t e^{-rs}[\alpha_s V^*(Z(s)+1) + \beta_s V^*(Z(s)-1) \\ &\quad - (\alpha_s + \beta_s + r)V^*(Z(s))]ds\} \\ &= V^*(x) + E_x\{\int_0^t e^{-rs}[(\Gamma\hat{V}^* - r\hat{V}^*)(Z(s)) - ((\alpha - \alpha_s)\Delta V^*(Z(s)+1) \\ &\quad - (\beta - \beta_s)\Delta V^*(Z(s))) - (q\beta - w\alpha)]ds\}. \end{aligned} \tag{3.15}$$

Putting (3.15) into (3.14),

$$\begin{aligned} V_t(x) = V^*(x) + E_x\{\int_0^t e^{-rs}[(\Gamma\hat{V}^* - r\hat{V}^*)(Z(s)) + (\beta - \beta_s)(\Delta V^*(Z(s)) - q) \\ + (\alpha - \alpha_s)(w - \Delta V^*(Z(s)+1))]ds\} \leq V^*(x). \end{aligned} \tag{3.16}$$

for $x \geq 0$ and $t \geq 0$ since $\Gamma\hat{V}^* - r\hat{V}^* \leq 0, w \leq \Delta V^* \leq q$, and $0 \leq \alpha_s \leq \alpha, 0 \leq \beta_s \leq \beta$ for $s \geq 0$ by Lemma 3.3, 3.4 and assumption (2.5). Noticing that

$$\lim_{t\to\infty} E_x\{\int_0^t e^{-rs}(q\beta_s - w\alpha_s)ds\} = F(x),$$

we have $F(x) \leq \lim_{t\to\infty} V_t(x) \leq V^*(x)$.

∎

The following proposition gives some monotonicity properties of the optimal inventory limit b^* as a function of other parameters.

Proposition 3.3. *The optimal inventory limit b^* increases as α, c or h decreases.*

Proof. Let

$$f(b, \alpha, c, p, h, r) \equiv \theta(b, 0) - \frac{w}{q}. \tag{3.17}$$

Note that b^* jumps only at the points where $f = 0$. According to condition (3.9), $f(b) = 0$ implies $b-1$ is optimal. If an increase in some parameter causes a decrease in f at the point where $f(b) = 0$, then $b - 1$ remains optimal to the right (i.e., $f(b-1) > 0$, and $f(b) < 0$), and b is optimal to the left (i.e., $f(b) > 0$ and $f(b = 1) < 0$). in this case b^* is a decreasing right continuous step function of the parameter. Conversely, if an increase in some parameter causes an increase in f, then b^* is an increasing left continuous step function of the parameter. However,

$$\frac{\partial f}{\partial c} < 0, \frac{\partial f}{\partial h} < 0, \frac{\partial f}{\partial \alpha} < 0 \text{ if } b > 1, \text{ and } \frac{\partial f}{\partial \alpha} = 0 \text{ if } b = 1.$$

∎

The intuition of proposition 3.3 is quite clear. An increase in production capacity α implies a higher frequency of production and a longer time for a product staying in stock. Therefore the firm prefers a lower inventory limit to modify this effect and avoid higher financial coat. A decrease in holding cost h or production cost c causes an increase in inventory limit simply because the firm can afford a higher inventory and then has more fulfilled demand. However the effect of price on b^* is ambiguous. The only remark we can make is that if the firm increases the price, then it tends to have a lower inventory limit when facing a more elastic market and a higher one when facing a less elastic market.

So far, a unique optimal policy is obtained for each specific capacity α and price p. Assuming that the optimal operating policy always follows after the design decisions are made, the expected profit is again a function of α and p, and an explicit form of it can be obtained. Theoretically, the optimal capacity can be determined by usual calculus. But it is not a trivial job. The difficulties comes from the fact that the first order derivative of the value function with respect to α or p is not continuous because of the discontinuity of b^* as a function of α or p. De Vany (1976) avoids the similar problem by approximating b, the balking value in his model, by a continuous differentiable function. We shall show that under a more rigorous mathematical treatment, the usual calculus and the marginal revenue-marginal cost interpretations can still be applied.

Let Π^* be the expected profit under the optimal barrier policy with capacity α and the price p. The relations (2.14) and (3.5) then imply

$$\begin{aligned}\Pi^*(x) &= V^*(x) - \frac{h}{r}x - C(\alpha)\\ &= \hat{V}^*(x) + \frac{q}{r}\beta - \frac{w}{r}\alpha - C(\alpha).\end{aligned} \tag{3.18}$$

To study the properties of Π^* as a function of α and p, it is sufficient to examine $\hat{V}^*$ since the other terms in (3.18) are assumed to be nice.

Lemma 3.6. *As a function of α, the V^* is continuous, strictly increasing, and differentiable except that*

$$\lim_{\alpha\downarrow\alpha_0} \frac{\partial V^*}{\partial\alpha} > \lim_{\alpha\uparrow\alpha_0} \frac{\partial V^*}{\partial\alpha}. \tag{3.19}$$

for each discontinuity point α_0 of b^*.

Proof. First note that as a function of α, the V^b (the value function with a *fixed* upper barrier b) is continuous and infinitely differentiable, and

$$V^* = \max\{V^1, V^2, \ldots, V^b, \ldots\}. \tag{3.20}$$

This fact immediately indicates the continuity of V^* with respect to α. The second assertion follows because if the upper bound of the production rate, α, increases to $\alpha+\delta$ ($\delta > 0$), then the firm

can do at least as well by feasibly employing the optimal policy with capacity α and in fact, it is strictly better off.

To prove the last assertion, notice the relation (3.20), that V^b is infinitely differentiable for any fixed b, and that b^* is a decreasing step function of α. So at each continuity point of b^*, the V^*, and hence Π^*, is differentiable. At each discontinuity point of b^*, the right and left derivatives exist and they are $\partial V^{b^*-1}/\partial\alpha$ and $\partial V^{b^*}/\partial\alpha$ respectively. Using the fact (3.10) and that b^* jumps only when $\theta(b^*,0) - w/q = 0$, we have that at each discontinuity point α_0 of b^*,

$$\begin{aligned}
\lim_{\alpha\downarrow\alpha_0}\frac{\partial V^*(x)}{\partial\alpha} - \lim_{\alpha\uparrow\alpha_0}\frac{\partial V^*(x)}{\partial\alpha} &= \frac{\partial}{\partial\alpha}[V^{b^*}(x) - V^{b^*-1}(x)] \\
&= \frac{\partial G}{\partial\alpha}[\theta(b^*,0) - \frac{w}{q}] + \frac{\partial\theta(b^*,0)}{\partial\alpha}G \\
&= G\frac{\partial\theta(b^*,0)}{\partial\alpha} < 0, \text{ for } b^* \geq 1,
\end{aligned}$$

since $G > 0$ and $\partial\theta(b,0)/\partial\alpha < 0$ where G is defined as in (3.10).

∎

Lemma 3.6 guarantees the existence of the optimal capacity α^* and its occurrence at a continuity point of $\partial\Pi/\partial\alpha$. Let us denote the expected total discounted actual sales and production by $\bar{B}$ and $\bar{A}$ respectively. Then

$$\bar{B} \equiv E_x[\int_0^\infty e^{-rt}dB(t)] = \frac{\beta}{r}\left[1 - \frac{(1-\rho_2^{-1})\rho_1^{-(b^*+1-x)} - (1-\rho_1^{-1})\rho_2^{-(b^*+1-x)}}{\rho_1^{-(b^*+1)} - \rho_2^{-(b^*+1)}}\right], \tag{3.21}$$

$$\bar{A} \equiv E_x[\int_0^\infty e^{-rt}dA(t)] = \frac{\alpha}{r}\left[1 - \frac{(1-\rho_2)\rho_1^{x+1} - (1-\rho_1)\rho_2^{x+1}}{\rho_1^{b^*+1} - \rho_2^{b^*+1}}\right]. \tag{3.22}$$

The conditions for the optimal price and the optimal capacity are as follows.

Proposition 3.4. *The optimal price p^* and the optimal capacity α^* satisfy the conditions:*

$$(p + \frac{h}{r})\frac{\partial\bar{B}}{\partial p} + \bar{B} - (c + \frac{h}{r})\frac{\partial\bar{A}}{\partial p} = 0, \tag{3.23}$$

$$(p + \frac{h}{r})\frac{\partial\bar{B}}{\partial\alpha} - (c + \frac{h}{r})\frac{\partial\bar{A}}{\partial\alpha} - \frac{\partial C}{\partial\alpha} = 0. \tag{3.24}$$

From (3.21) and (3.22), we see that $\bar{B}$ and $\bar{A}$ are functions of p only through $\beta(p)$. it can be shown that $\bar{A}$ is an increasing function of β, hence, a decreasing function of p, and $\bar{B}$ is an increasing function of β only when β is large relative to α,hence, a decreasing function of p. However, the necessary condition for optimality (3.23) shows that the optimal p^* is always located at the decreasing stretch of $\bar{B}$ as a function of p. Because otherwise, the left side of (3.23) would be strictly positive, and then there would be room for improvement by changing p. Recall that $\beta(p)$ is

the mean rate of potential demand whereas $\bar{B}(p)$ is the expected total discounted demands which are actually fulfilled. For this reason, we refer to $\bar{B}(p)$ as the *effective demand* function defined on the decreasing portion and $\beta(p)$ as the *potential demand* functions. Denote the *price elasticity of effective demand* by $\bar{\epsilon}$ and that of potential demand by ϵ, i.e.,

$$\epsilon \equiv \frac{d\beta}{dp} \cdot \frac{p}{\beta}, \text{ and } \bar{\epsilon} \equiv \frac{\partial \bar{B}}{\partial p} \cdot \frac{p}{\bar{B}}.$$

Proposition 3.5. *If $\beta \geq \alpha$, then the price elasticity of effective demand $\bar{\epsilon}$ is smaller than the potential elasticity ϵ in the absolute value.*

Proof. Show only for zero initial inventory. Let

$$\bar{L} \equiv \frac{(1-\rho_2^{-1})\rho_1^{-(h+1)} - (1-\rho_1^{-1})\rho_2^{-(h+1)}}{\rho_1^{-(h+1)} - \rho_2^{-(h+1)}}.$$

Equation (3.21) then becomes $\bar{B} = \frac{\beta}{r}(1-\bar{L})$, and

$$\frac{\partial \bar{B}}{\partial p} = \frac{1}{r}(1-\bar{L})\frac{d\beta}{dp} - \frac{\beta}{r}\cdot\frac{\partial \bar{L}}{\partial p} = \frac{d\beta}{dp}\cdot\frac{\bar{B}}{\beta} - \frac{\beta}{r}\cdot\frac{\partial \bar{L}}{\partial \beta}\cdot\frac{d\beta}{dp}$$

which implies that

$$|\bar{\epsilon}| = |\epsilon| + \frac{\beta p}{r\bar{B}}\cdot\frac{\partial \bar{L}}{\partial \beta}\cdot\frac{d\beta}{dp} < |\epsilon|,$$

since $\partial\bar{L}/\partial\beta > 0$ if $\beta \geq \alpha$ and $d\beta/dp < 0$.

∎

To compare with the deterministic theory, we can rewrite condition (3.23) as

$$p(1+\frac{1}{\bar{\epsilon}}) = \frac{c(\partial\bar{A}/\partial p) + \frac{h}{r}((\partial\bar{A}/\partial p) - (\partial\bar{B}/\partial p))}{\partial\bar{B}/\partial p}$$

This is analogous to the traditional monopoly result in that marginal revenue equals marginal cost, price is set in the elastic portion of the demand function, and is a markup over marginal cost, i.e.,

$$p(1+\frac{1}{\epsilon}) = c. \tag{3.25}$$

However the relevant marginal cost in the stochastic model is a cost of total discounted actual sales instead of that of output and is composed of two parts, production cost and inventory carrying cost.

Equation (3.24) can be viewed as a *long-run* condition since it determines the production capacity of the firm. It indicates that capacity is expanded to the point where the expected

marginal revenue achieved through reduction of the inventory limit equals the marginal cost of capacity, and can be rewritten as

$$p = \frac{c(\partial \bar{A}/\partial \alpha) + \frac{h}{r}((\partial \bar{A}/\partial \alpha) - (\partial \bar{B}/\partial \alpha)) + (\partial C/\partial \alpha)}{\partial \bar{B}/\partial \alpha}. \tag{3.26}$$

The firm sets the price equal to the long-run marginal cost of total discounted actual sales. The numerator of the right side in equation (3.26) is the present value of the full incremental cost of increasing capacity which consists of the short-term marginal cost of output times the increase in average total production stream induced by greater capacity, the increase in average total inventory holding cost due to an increase in capacity, plus the marginal cost of capacity building. This full incremental cost of capacity is multiplied by $1/(\partial \bar{B}/\partial \alpha)$, the increment to capacity required to induce a unit increase in average total sales.

Some interesting remarks can be drawn from our model which copes with a stochastic multi-period situation. First, it is necessary to formulate actual versus potential production and demand, and to introduce a buffer stock if possible. Secondly, with stochastic variability, production does not always meet demand even in the sense of expected value, that is α^* does not necessarily equal β. As a matter of fact, the firm always has excess capacity in response to a stochastic situation in the sense that α^* always exceeds the mean rate of fulfilled demand. Finally, seeking a stochastic theory of the firm, one inevitably arrives at a dynamic model of the firm where the decision process is split into long-run design decisions and short-run operating decisions, because of the central role of the inventories in responding to stochastic variability.

4. Dynamic Pricing

We are now in a position to solve the basic model formulated in Section 2 with the further generality that dynamic pricing is allowed.

In the investigation of the optimal operating production and pricing decisions, the explicit calculation as done in the previous section is no longer easy if not impossible and our discussion will be facilitated by considering a class of problems known as semi-Markov decision processes. The existence of a finite stationary solution can be shown by the contraction mapping fixed point theorem or the general theory of semi-Markov decision processes (see, e.g., [7], [13], [14] and [21]). Also we shall leave the completeness of Markovian controls aside and omit most of the proofs of the results parallel to those in the previous section. The interested readers are referred to [16].

For notation simplicity, we still denote the optimal value functions by V and Π. In the context of an semi-Markov decision processes, the recursive equations of the firm's problem can be specified as follows

$$rV(x) = \max_{\alpha' \in [0,\alpha], \beta' \in [0, \beta 1_{(0,\infty)}(x)]} \{\beta'[p(\beta') + \frac{h}{r} - \Delta V(x)] + \alpha'[\Delta V(x+1) - (c + \frac{h}{r})]\}, \tag{4.1}$$

where $\Delta V(x) \equiv V(x) - V(x-1)$.

Lemma 4.1. *For fixed α, the value function $V(\cdot)$ is strictly increasing and concave.*

It is easy to see that for each x, if $\alpha(x)$ solves (4.1), then

$$\alpha(x) = \begin{cases} \alpha, & \text{if } \Delta V(x+1) > c + \frac{h}{r} \text{ or, equivalently, } \Delta\Pi(x+1) > c; \\ 0, & \text{if } \Delta V(x+1) \leq c + \frac{h}{r} \text{ or, equivalently, } \Delta\Pi(x+1) \leq c, \end{cases}$$

noticing that $\Delta\Pi(x) = \Delta V(x) - \frac{h}{r}$. Suppose b is the smallest x such that $\Delta\Pi(x+1) \leq c$, then $\alpha(x) = \alpha$ for $0 \leq x \leq b-1$, and $\alpha(b) = 0$. That means for fixed α, the operating policy is still a barrier policy with a inventory limit b. Lemma 4.1 implies that b is uniquely determined by the following conditions

$$\Delta\Pi(b) > c, \text{ and } \Delta\Pi(b+1) \leq c.$$

The pricing policy can be solved by examining (4.1) in a similar fashion. The solution $\beta(x)$ should satisfy

$$p + \beta\frac{dp}{d\beta} - \Delta\Pi(x) = 0, \text{ for } 1 \leq b.$$

In the similar way of proving Proposition 3.2, we have

Proposition 4.1. *For fixed α, the optimal operating policy is a barrier policy with inventory limit b, that is*

$$\alpha_t = \alpha 1_{[0,b)}(Z(t-)), \tag{4.2}$$

where b is uniquely determined so that

$$\Delta\Pi(b) > c, \text{ and } \Delta\Pi(b+1) \leq c. \tag{4.3}$$

The optimal pricing policy is

$$p(\beta_t) = \sum_{x=1}^{b} p(\beta(x))1_{\{x\}}(Z(t-)). \tag{4.4}$$

and $\beta(x)$ is determined by the following short-run conditions:

$$p + \beta\frac{dp}{d\beta} - \Delta\Pi(x) = 0, \text{ for } 1 \leq x \leq b. \tag{4.5}$$

The implication of condition (4.3) is that the firm will produce to stock at its full capacity to a limit where the short-term marginal cost of production will exceed the present value of the total future profit increment of an additional unit of output at the moment. Regarding the optimal inventory limit b which solely determines the optimal operating policy, similar comparative static results hold in the dynamic pricing case.

Proposition 4.2. *The optimal inventory limit b decreases as α, c, or h decreases.*

On the other hand, equation (4.5) can be written as

$$p(1 + \frac{1}{\epsilon}) = \Delta\Pi(x). \tag{4.6}$$

where ϵ is the price elasticity of potential demand defined as in the previous section. Here, $\Pi(x)$ is the total expected future profit given that there are presently x units of product on hand. Suppose one unit is sold at the moment, then the present value of net profit for this unit is $p+\Pi(x-1)-\Pi(x)$. Therefore $\Delta\Pi(x)$ can be interpreted as the marginal cost of increase in sales by one unit. Equation (4.6) is analogous to the traditional monopoly result (3.25). However it is noteworthy that instead the actual production unit cost, c, the correct marginal cost in a stochastic model should be the unit cost of actual sales, $\Delta\Pi(x)$, the increase in cost of selling one unit out of stock leaving the capacity, the optimal operating and the optimal pricing policies unchanged for the future.

Proposition 4.3. *The price decreases as the inventory level increases and is always higher than that under certainty, i.e.,*

$$p(\beta(1)) > p(\beta(2)) > \ldots > p(\beta(b)) \geq p^d. \tag{4.7}$$

where p^d is traditional monopoly price determined in (3.25).

Proof. Lemma 4.1 shows that $\Delta V(x)$, hence $\Delta\Pi(x)$, is decreasing in x. And the second order condition for optimality says that $p(1+1/\epsilon)$ is decreasing in β. Condition (4.5) which determines $\beta(x)$ then implies that $\beta(x)$ increases as x increases, or $p(\beta(x))$ decreases as x increases. Finally, Lemma 4.1 and condition (4.3) imply that

$$\Delta\Pi(x) \geq \Delta\Pi(b) > c, \text{ for } 1 \leq x \leq b.$$

It becomes obvious that $p(\beta(x)) > p^d$ for $1 \leq x \leq b$.

∎

Intuitively one can think that the more products pile up in inventory, the more incentive the firm has to lower the price and encourage demand for the sake of reducing its holding cost. This line of thinking may be misleading since following it, one may gather that, with zero inventory, the firm would have a best situation and simulate the deterministic pricing decision. However, the fact is that, because of stochastic variability, the more product the firm has on hand, the better off it is. This can be seen from the fact that $\Pi(x) - \Pi(x-1) > c > 0$ for $0 \leq x \leq b$. Therefore, with lower stock on hand, the firm sets higher prices to discourage demand in order to increase the stock lever anticipating higher profit in the future. This procedure continues to the point at which the

inventory level reaches b and the marginal cost of actual sales $\Pi(b) - \Pi(b-1)$ is closest to that in the deterministic case, c. It is at this point that the firm ceases production optimally and simulates the deterministic pricing decision. In sum, the firm transfers the cost of uncertainty to consumers by raising prices.

Finally, we derive the condition for the optimal capacity decision. The value function under the optimal operating and pricing policies V possesses the same properties as in Lemma 3.6.

Lemma 4.2. *As a function of α, the V is continuous, strictly increasing, and differentiable except that*

$$\lim_{\alpha \downarrow \alpha_0} \frac{\partial V}{\partial \alpha} > \lim_{\alpha \uparrow \alpha_0} \frac{\partial V}{\partial \alpha}.$$

for each discontinuity point α_0 of b.

Still as before, we define

$$\bar{B} \equiv E_x \left\{ \int_0^\infty e^{-rt} dB(t) \right\} \text{ and } \bar{A} \equiv E_x \left\{ \int_0^\infty e^{-rt} dA(t) \right\}.$$

In addition, we let TR be the present value of the expected total revenue,

$$TR \equiv E_x \left\{ \int_0^\infty e^{-rt} p(\beta_t) dB(t) \right\}.$$

Proposition 4.4. *The optimal capacity α^* satisfies the long-run condition*

$$\frac{\partial TR}{\partial \alpha} = c \frac{\partial \bar{A}}{\partial \alpha} + \frac{h}{r}\left(\frac{\partial \bar{A}}{\partial \alpha} - \frac{\partial \bar{B}}{\partial \alpha}\right) + \frac{\partial C}{\partial \alpha}. \tag{4.8}$$

Condition (4.8) says that the firm expands its capacity to the point where the expected marginal revenue achieved through reduction of the inventory limit equals the marginal cost of capacity, which consists of production cost, holding cost and capacity building cost, induced by the increase in the potential production rate.

5. Learning Effects

Learning effects can be introduced into the model of the preceding section. In most of the literature, learning effects are introduced by the assumption that unit cost declines with the accumulated output or production (see, e.g., [1], [12] and [23]). However, we attempt to introduce learning effects into the current model in a more direct way in which productivity increases with cumulative production.

Let $\alpha\gamma(A(t-))$ be the upper bound of the production rate that the firm can achieve at time t, where $\gamma(a)$ is increasing and concave in a, $\gamma(0) > 0$, and $\gamma(a) \to 1$ as $a \to \infty$. So $\alpha\gamma(0)$ is the capacity achievable at time zero, and α is the capacity achievable in the long run. The economic justification is simple. The firm builds up an ideal capacity level α at the beginning,

and its full capacity production rate gets greater and greater approaching the ideal capacity as the management and the labor become more and more sophisticated through actually producing.

With learning effects introduced this way, the modification in the basic model formulated in Section 2 is that the feasibility constraint (2.5) should be altered to be

$$0 \leq \alpha_t \leq \alpha\gamma(A(t-)), \beta_t \geq 0 \text{ and } \beta_t \text{ are bounded, for all } t \geq 0. \tag{5.1}$$

The firm's goal is to choose a pair of control processes (α_t, β_t) and a long-run achievable capacity α to maximize expected profit Π providing the learning curve $\gamma(\cdot)$.

For a fixed α, in order to investigate the optimal operating and pricing policies in the presence of learning effect, we need to expand the state space to include cumulative production. Each state (x, a) is a pair of non-negative integers with the first element representing the inventory level and the second representing the cumulative output. The system is said to be in state (x, a) at time t if the firm has x units of product in stock and has produced a units up to time t, or say, it has full capacity production rate $\alpha\gamma(a)$ to start with from time t on. Denote the optimal value functions with learning by $\Pi(x, a)$ and $V(x, a)$ satisfying relation (2.14), i.e.,

$$V(x,a) = \max_{0 \leq \alpha_t \leq \alpha\gamma(a+A(t-)), 0 \leq \beta_t \leq \beta 1_{(0,\infty)}(Z(t-))} E_{x,a}\left\{\int_0^\infty e^{-rt}[(p(\beta_t) + \frac{h}{r})dB(t) - (c + \frac{h}{r})dA(t)]\right\}. \tag{5.2}$$

Let $V^\alpha(x)$ be the value function with capacity α in the absence of learning effects as in the previous section. Simple observations indicate that

Lemma 5.1. *$V(x, a)$ increases and converges to $V^\alpha(x)$ as a increases to infinity.*

Proof. Starting with $(x, a+1)$, the firm would do at least as well by following the optimal policies with starting state (x, a). This is feasible since $\gamma(a+1+\cdot) > \gamma(a+\cdot)$. So $V(x, a+1) \geq V(x, a)$. Similar argument implies that

$$V^{\alpha\gamma(a)}(x) \leq V(x, a) \leq V^\alpha(x) \text{ since } \alpha\gamma(a) \leq \alpha\gamma(a+\cdot) \leq \alpha.$$

Letting $a \to \infty$, hence $\alpha\gamma(a) \to \alpha$, we have

$$0 \leq \lim_{a \uparrow \infty}[V^\alpha(x) - V(x, a)] \leq \lim_{a \uparrow \infty}[V^\alpha(x) - V^{\alpha\gamma(a)}(x)] = 0$$

since $V^\alpha(x)$ is continuous in α as shown in Lemma 3.6.

∎

The recursive equations that $V(x, a)$'s satisfy can be specified as

$$\begin{aligned} rV(x,a) = \max_{\alpha' \in [0, \alpha\gamma(a)], \beta' \in [0, \beta 1_{(0,\infty)}(x)]} &\{\beta'[p(\beta') + \frac{h}{r} - (V(x, a) - V(x-1, a))] \\ &+ \alpha'[(V(x+1, a+1) - V(x, a)) - (c + \frac{h}{r})]\}. \end{aligned} \tag{5.3}$$

Lemma 5.2. *For fixed α and $\gamma(\cdot)$, the optimal value function $V(\cdot, a)$ is strictly increasing and concave for every $a \geq 0$.*

Proof. By double induction. For details see Appendix C in [16].

∎

Examining recursive equation (5.3), we have

Proposition 5.1. *For fixed long-run achievable capacity α and learning curve $\gamma(\cdot)$, the optimal production policy is a barrier policy with inventory limit $b(\cdot)$, a decreasing function of cumulative production, that is*

$$\alpha_t = \sum_{a=0}^{\infty} \sum_{x=0}^{b(a)} \alpha \cdot \gamma(a) \cdot 1_{\{x,a\}}(Z(t-), A(t-)). \tag{5.4}$$

where $b(a)$ is uniquely determined so that

$$\Pi(b, a) - \Pi(b-1, a-1) > c, \text{ and } \Pi(b+1, a) - \Pi(b, a-1) \leq c. \tag{5.5}$$

The optimal pricing policy is

$$p(\beta_t) = \sum_{a=0}^{\infty} \sum_{x=1}^{b(a)} p(\beta(x, a)) \cdot 1_{\{x,a\}}(Z(t-), A(t-)). \tag{5.6}$$

where $\beta(x, a)$ is determined by the following short-run condition

$$p + \beta \frac{dp}{d\beta} - [\Pi(x, a) - \Pi(x-1, a)] = 0. \tag{5.7}$$

for each $x \in [1, b(a)], a \geq 0$.

Regarding the operating policy, we note that the upper barrier is no longer a single number as before. As a function of t, the optimal inventory limit $b(A(t-))$ is stochastic since the accumulated production is stochastic. As a function of the cumulative actual production, $b(\cdot)$ is a deterministic function and is defined by the condition (5.5). This again indicates that the firm will produce to stock at its full capacity achievable up to a limit where the short-run marginal cost, c, will exceed the present value of the total profit increment induced by an additional unit of output. It can be shown that as a increases to infinity, the optimal inventory limit $b(a)$ decreases to b^{α}, the optimal inventory limit with capacity α and without learning effects. That is, the firm lowers the inventory limit gradually as its production experience grows. Also, the downward-sloping upper barrier $b(\cdot)$ will be shift upwards as h, c, or α decreases.

To investigate the properties of the optimal pricing policy with learning, rewrite condition (5.7) as

$$p(1 + \frac{1}{\epsilon}) = \Pi(x, a) - \Pi(x-1, a), \tag{5.8}$$

where ϵ is the price elasticity of potential demand as before. The difference, $\Pi(x, a) - \Pi(x-1, a)$, is the present value of the total future profit reduction due to taking one unit out of stock. It can be interpreted as the (opportunity) marginal cost of increase in sales by one unit. By lemma 5.2, $\Pi(x, a) - \Pi(x-1, a)$ is decreasing for any fixed $a \geq 0$. This implies

$$p(\beta(x, a)) < p(\beta(x-1, a)), \text{ for } 1 \leq x \leq b(a) \text{ and } a \geq 0.$$

Therefore the effect, that in a stochastic environment the firm with lower stock on hand has the tendency to set a higher price to discourage demand and hence to achieve higher inventory level anticipating higher future profit, still exists as that in the absence of learning. However, this tendency is moderated by the learning effects. To see this, note that

$$\Pi(x, a) - \Pi(x-1, a-1) = [\Pi(x, a) - \Pi(x-1, a)] + [\Pi(x-1, a) - \Pi(x-1, a-1)]. \qquad (5.9)$$

Both terms on the right side of (5.9) are positive. The first term is the marginal cost of actual sales which determines the optimal prices, while the second term reflects the profit gain by learning. The firm chooses the optimal inventory limit $b(a)$ by jointly considering these two effects (see condition (5.5) in Proposition 5.1). Suppose b is so chosen that

$$\Pi(b, a) - \Pi(b-1, a-1) = c.$$

Then

$$\Pi(b, a) - \Pi(b-1, a) = [\Pi(b, a) - \Pi(b-1, a-1)] - [\Pi(b-1, a) - \Pi(b-1, a-1)] < c,$$

and hence,

$$p(\beta(b, a)) < p^d,$$

where p^d is the monopoly price under certainty satisfying (3.25). The interesting point here is that the firm with stochastic variability tends to transfer the cost of uncertainty to consumers, but the presence of learning effects moderates this transfer. Therefore the monopoly price with uncertainty as well as learning effects does not always dominate the monopoly price under certainty. When the firm become more and more mature, the effect of learning is weaker and weaker, and this moderation diminished gradually.

Proposition 5.2. *The price decreases as the inventory level increases for any fixed level of cumulative production but does not always dominates that under certainty.*

Similar to the case without learning effects, the long-run condition equating the long-term marginal revenue and marginal cost of capacity determines the optimal capacity α^*.

Proposition 5.3. *The optimal capacity α^* satisfies the condition*

$$\frac{\partial TR}{\partial \alpha} = c\frac{\partial \bar{A}}{\partial \alpha} + \frac{h}{r}\left(\frac{\partial \bar{A}}{\partial \alpha} - \frac{\partial \bar{B}}{\partial \alpha}\right) + \frac{\partial C}{\partial \alpha},$$

where $TR \equiv E\{\int_0^\infty e^{-rt} p(\beta_t) dB(t)\}$, $\bar{A} \equiv E\{\int_0^\infty e^{-rt} dA(t)\}$, *and* $\bar{B} \equiv E\{\int_0^\infty e^{-rt} dB(t)\}$.

6. Concluding Remarks

The article introduces a model of production firms by assuming that cumulative production and cumulative demand are two counting processes with random intensity parameterized by production capacity and price respectively. The formulation captures many important characteristics of production firms, such as the distinction among production capacity, actual production rate, demand rate, and actual sales rate; the distinction between static design decisions (long-run decisions) and dynamic operating decisions (short-run decisions), etc. In particular, one obtains a fundamentally dynamic theory, with inventory tying together production decisions and pricing decisions at different points in time.

This is a natural generalization of the classical (deterministic) model of the firm. In fact, it can be shown that a sequence of the Poisson-Poisson problems formulated in Section 2 will converge to a deterministic model as certain parameters approach certain limits, keeping the variance approaching zero. On the other hand, close to the special case described in Section 3 is a diffusion model of inventory and production control studied by Harrison in [11]. There the difference of cumulative potential input and cumulative potential demand is modeled by a Brownian motion (with general drift and variance parameters), and the optimal policy (involving a single critical number b^*) is very simple. However, in the diffusion model, the design decisions are lumped into a single drift term, which are explicitly accounted for in our model. Nevertheless, Li shows in [16] that this diffusion model represents another limit of our basic model as certain parameters approach critical values. The limit result helps one to better understand conditions under which the diffusion model applies, and justifies a very tractable approximation for general additive process formulation.

There are many other interesting questions, both mathematical and economic, that might be explored in the continuation of this work. Our model can be generalized to the formulation where cumulative production $\hat{A}$ and cumulative potential demand $\hat{B}$ are arbitrary additive processes. Suppose, for example, the primitive processes are compound Poisson with absolutely continuous jump size distributions. The optimal operating policy would still be a barrier policy with one critical parameter b^*. But, it is not clear how families of potential processes would be "parameterized" by basic price and capacity decisions. This problem is not surprising - in real life there may be many different "kinds of capacity", and one can also abstract different "kinds of business" with different market strategies. Secondly, the models in this study seem appropriate in the markets

characterized by non-durable goods since there is no trend in the demand pattern. They can be generalized to include the durable good markets by assuming that the demand rate is a function of price as well as cumulative demand. Thirdly, the variable cost of production is assumed to be linear, and there is no cost associated with varying the production rate. Certainly, we can have more general cost structure. As a first step, we can associate a set-up cost with each restart of production, and this does not create much difficulty at least in the computations as we carry out in Section 3.

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